Linking electroweak and gravitational generators

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Using complexified quaternions, an intriguing link between generators of two different and surprisingly *commuting* four-dimensional representations of the $\mathrm{SU}(2) \times \mathrm{U}(1)$ Lie group, and generators of two four-dimensional spin $\frac{1}{2}$ representations of the $\mathrm{Spin}(3,1)$ Lie group is established: the former generators completely determine the latter ones, and cross-combined they constitute two different, but closely related, four-dimensional representations of $\mathrm{Spin}(3,1) \times \mathrm{SU}(2) \times \mathrm{U}(1)$. These representations are used to construct a $\mathrm{Spin}(3,1) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ gauge invariant Lagrangian, containing two four-spinors consisting not as usual of Weyl two-spinors of opposite helicity and equal weak isospin, but instead of Weyl two-spinors of opposite weak isospin and equal helicity, a construction which arises naturally from the mathematical formalism itself. A possible future generalization, using complexified octonions, is discussed.

I. INTRODUCTION AND MAIN RESULT

The quest for unification of the fundamental forces of Nature began with the unification of electricity and magnetism, by Maxwell, resulting in the electromagnetic force [1]. A century later this force was united with the weak nuclear force, resulting in the electroweak force [2]. Even though there have been promising theoretical propositions of unification of this force with the strong nuclear force, none of these have been experimentally verified. Gravity [3] is in its very own category, stubbornly refusing to join the quantum-party of unification, the currently most promosing, though highly speculative, theoretical proposition being string/M-theory [4, 5].

This article makes no claim of any unification. More modestly, it is the purpose of this article to point out, and utilize, an intriguing link between generators of two different and surprisingly *commuting*, see Eq. (5), four-dimensional representations of the $\mathrm{SU}(2) \times \mathrm{U}(1)$ Lie group (relevant for the electroweak force), and generators of two four-dimensional spin $\frac{1}{2}$ representations of the $\mathrm{Spin}(3,1)$ Lie group (relevant for the interaction, described in terms of a minimal spin connection, of the gravitational field and spinor fields): the former generators completely determine the latter ones, see Eq. (8), and cross-combined they constitute two different, but closely related (by complex conjugation), four-dimensional representations of $\mathrm{Spin}(3,1) \times \mathrm{SU}(2) \times \mathrm{U}(1)$, see Sec. III C. Mathematically, this link between generators is (most directly) established using complexified quaternions, also called biquaternions. Physically, this link is established by grouping together in two four-spinors not the usual Weyl two-spinors of opposite helicity and equal weak isospin, but instead Weyl two-spinors of opposite weak isospin and equal helicity, a construction which arises naturally from the mathematical formalism itself. The main result of the article is the $\mathrm{Spin}(3,1) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ gauge invariant Lagrangian, Eq. (14).

No formal introduction to the quaternions will be given, the reader kindly being referred to for instance Refs. [6, 7]. Neither will any formal introduction to the (complexified) octonions, or the even more general composition algebras, be given, the reader kindly being referred to the literature: For short reviews of the octonions, see Refs. [8, 9, 10, 11]. For a comprehensive review of the octonions, see Ref. [12]. For a monograph on octonions and other nonassociative algebras, see Ref. [13]. In particularly, for a monograph on composition algebras, a class to which both the complex quaternions and complex octonions belong (note that they are not division algebras, even though the quaternions and the octonions themselves are), see Ref. [14].

The paper is organized as follows: Sec. II introduces the necessary notation and conventions used. Sec. III sets up the main machinery needed. Sec. IV contains the main result of the paper; the $\mathrm{Spin}(3,1) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ gauge invariant Lagrangian, Eq. (14). Sec. V discusses various notable features of this Lagrangian, and points to a future generalization, using complexified octonions. There are two appendices: Appendix A contains various useful identities valid for any composition algebra, a class to which both the complexified quaternions and complexified octonions belong. Appendix B contains the proofs of most of the assertions of Sec. III.

II. NOTATION AND CONVENTIONS

The set of complexified quaternions is denoted $\mathbb{C} \otimes \mathbb{H}$, equal to $\mathbb{H} \otimes \mathbb{C}$ because the complex numbers $\mathbb{C}$ and the quaternions $\mathbb{H}$ are assumed to commute. The imaginary unit of $\mathbb{C}$ is denoted i, obeying $\mathrm{i}^2 = -1$, of course. The imaginary units of $\mathbb{H}$ are denoted $\mathrm{e}_i = (\mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3)$, obeying $\mathrm{e}_i \mathrm{e}_j = -\delta_{ij} + \varepsilon_{ij}{}^{l} \mathrm{e}_l$, where ε_{ijk} is the Levi-Civita symbol with $\varepsilon_{123} = +1$. The basis for $\mathbb{C} \otimes \mathbb{H}$ (over $\mathbb{C}$) is taken as $\mathrm{e}_a = (\mathrm{e}_0, \mathrm{e}_i) = (\mathrm{i}, \mathrm{e}_i)$.

Latin indices from the beginning of the alphabet run from 0 to 3, and are raised and lowered with η^{ab} and η_{ab}, respectively, η_{ab} being the Minkowski metric. Latin

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indices from the middle of the alphabet, beginning at i, run from 1 to 3, and are raised and lowered with δ^{ij} and δ_{ij}, respectively. Greek indices run from 0 to 3, and are raised and lowered with $g^{\mu\nu}$ and $g_{\mu\nu}$, respectively, $g_{\mu\nu}$ being the metric of curved spacetime. The Einstein summation convention is adhered to throughout.

Let $c^a \in \mathbb{C}$. Complex conjugation is the involution $\cdot^* : \mathbb{C} \otimes \mathbb{H} \to \mathbb{C}^* \otimes \mathbb{H}$ defined by $(c^a \mathrm{e}_a)^* = -\left(c^0\right)^* \mathrm{e}_0 + \left(c^i\right)^* \mathrm{e}_i$, and quaternionic conjugation is the involution $\bar{\cdot} : \mathbb{C} \otimes \mathbb{H} \to \mathbb{C} \otimes \overline{\mathbb{H}}$ defined by $\overline{c^a \mathrm{e}_a} = c^0 \mathrm{e}_0 - c^i \mathrm{e}_i$. Note that $\overline{\mathrm{e}}_a^* = -\mathrm{e}_a$.

The bilinear inner product $\langle \cdot, \cdot \rangle : (\mathbb{C} \otimes \mathbb{H})^2 \to \mathbb{C}$ is defined by

$$2 \langle x, y \rangle = x\overline{y} + y\overline{x} \equiv \overline{x}y + \overline{y}x. \tag{1}$$

Note that $\langle \mathrm{e}_a, \mathrm{e}_b \rangle = \eta_{ab}$.

The set of n-dimensional square matrices over some field $\mathbb{F}$ is denoted $\mathrm{M}(n, \mathbb{F})$. The n-dimensional identity matrix is denoted $\mathbf{1}_n$, and the n-dimensional matrix with zero entries only is denoted $\mathbf{0}_n$. The socalled 'eta-transpose' $\cdot^\eta : \mathrm{M}(4, \mathbb{C}) \to \mathrm{M}(4, \mathbb{C})$ is defined by

$$\mathbf{A}^\eta = \eta \mathbf{A}^{\mathrm{T}} \eta,$$

where η is the Minkowski metric as a matrix; $(\eta)^a{}_b = \eta_{ab} = \eta^{ab}$. In terms of this 'eta-transpose', the two (anti)commutator-like [even though they do not have all the properties of the usual (anti)commutator] brackets $[\cdot, \cdot]_{\eta\pm} : \mathrm{M}(4, \mathbb{C}) \to \mathrm{M}(4, \mathbb{C})$, are defined by

$$[\mathbf{A}, \mathbf{B}]_{\eta\pm} = \mathbf{A}^\eta \mathbf{B} \pm \mathbf{B}^\eta \mathbf{A}.$$

III. SETUP

For analytical proofs of various assertions of this section, see Appendix B.

A. Generators of $\mathrm{SU}(2) \times \mathrm{U}(1)$

Define the matrices $\mathbf{\Gamma}_{L|a}, \mathbf{\Gamma}_{R|a} \in \mathrm{M}(4, \mathbb{C})$ by

$$\left(\mathbf{\Gamma}_{L|a}\right)_{cd} = \langle \mathrm{e}_c, \mathrm{e}_a \mathrm{e}_d \rangle, \tag{2a}$$
$$\left(\mathbf{\Gamma}_{R|a}\right)_{cd} = \langle \mathrm{e}_c, \mathrm{e}_d \mathrm{e}_a \rangle, \tag{2b}$$

where L and R refer to whether e_a is multiplied from the left or the right. They obey (note the mixed positions of L and R)

$$\mathbf{\Gamma}^*_{L|a} = -\mathbf{\Gamma}_{R|a}, \quad \mathbf{\Gamma}^*_{R|a} = -\mathbf{\Gamma}_{L|a}, \tag{3a}$$
$$\mathbf{\Gamma}^{\mathrm{T}}_{L|a} = +\mathbf{\Gamma}_{R|a}, \quad \mathbf{\Gamma}^{\mathrm{T}}_{R|a} = +\mathbf{\Gamma}_{L|a}, \tag{3b}$$
$$\mathbf{\Gamma}^\dagger_{L|a} = -\mathbf{\Gamma}_{L|a}, \quad \mathbf{\Gamma}^\dagger_{R|a} = -\mathbf{\Gamma}_{R|a}. \tag{3c}$$

Also, they obey the Lie algebra (where X denotes either L or R, a shorthand notation frequently used below)

$$-2\varepsilon_{ij}{}^k \mathbf{\Gamma}_{L|k} = \left[\mathbf{\Gamma}_{L|i}, \mathbf{\Gamma}_{L|j}\right], \tag{4a}$$
$$+2\varepsilon_{ij}{}^k \mathbf{\Gamma}_{R|k} = \left[\mathbf{\Gamma}_{R|i}, \mathbf{\Gamma}_{R|j}\right], \tag{4b}$$
$$\mathbf{0}_4 = \left[\mathbf{\Gamma}_{X|0}, \mathbf{\Gamma}_{X|i}\right], \tag{4c}$$

so that $\mathrm{i}\mathbf{\Gamma}_{L|a}$ and $\mathrm{i}\mathbf{\Gamma}_{R|a}$ (note the i's) each constitute hermitian generators of two different, but closely related (by complex conjugation), four-dimensional representations of $\mathrm{SU}(2) \times \mathrm{U}(1)$. In fact, due to the surprising relation (which originally prompted this research)

$$\mathbf{0}_4 = \left[\mathbf{\Gamma}_{L|a}, \mathbf{\Gamma}_{R|b}\right], \tag{5}$$

they constitute two *commuting* representations.

Furthermore, they obey the anticommutator-like relation

$$2\eta_{ab}\mathbf{1}_4 = \left[\mathbf{\Gamma}_{X|a}, \mathbf{\Gamma}_{X|b}\right]_{\eta+} \equiv \mathbf{\Gamma}^\eta_{X|a}\mathbf{\Gamma}_{X|b} + \mathbf{\Gamma}^\eta_{X|b}\mathbf{\Gamma}_{X|a}, \tag{6}$$

which is the reason for the choice of '$\mathbf{\Gamma}$' as the designating letter, $\mathbf{\Gamma}_{L|a}$ and $\mathbf{\Gamma}_{R|a}$ being reminiscent of the usual Clifford gamma matrices.

B. Generators of $\mathrm{Spin}(3,1)$

Define the matrices $\mathbf{\Sigma}_{L|ab}, \mathbf{\Sigma}_{R|ab} \in \mathrm{M}(4, \mathbb{C})$ by

$$4\mathrm{i}\left(\mathbf{\Sigma}_{L|ab}\right)_{cd} = \langle \mathrm{e}_a\mathrm{e}_c, \mathrm{e}_b\mathrm{e}_d \rangle - \langle \mathrm{e}_a\mathrm{e}_d, \mathrm{e}_b\mathrm{e}_c \rangle, \tag{7a}$$
$$4\mathrm{i}\left(\mathbf{\Sigma}_{R|ab}\right)_{cd} = \langle \mathrm{e}_c\mathrm{e}_a, \mathrm{e}_d\mathrm{e}_b \rangle - \langle \mathrm{e}_d\mathrm{e}_a, \mathrm{e}_c\mathrm{e}_b \rangle, \tag{7b}$$

where L and R refer to whether e_a and e_b are multiplied from the left or the right. They are related to $\mathbf{\Gamma}_{L|a}$ and $\mathbf{\Gamma}_{L|b}$ by the commutator-like relations

$$4\mathrm{i}\mathbf{\Sigma}_{X|ab} = \left[\mathbf{\Gamma}_{X|a}, \mathbf{\Gamma}_{X|b}\right]_{\eta-} \equiv \mathbf{\Gamma}^\eta_{X|a}\mathbf{\Gamma}_{X|b} - \mathbf{\Gamma}^\eta_{X|b}\mathbf{\Gamma}_{X|a}. \tag{8}$$

They obey (note the mixed positions of L and R)

$$\mathbf{\Sigma}^*_{L|ab} = -\mathbf{\Sigma}_{R|ab}, \quad \mathbf{\Sigma}^*_{R|ab} = -\mathbf{\Sigma}_{L|ab}, \tag{9a}$$
$$\mathbf{\Sigma}^{\mathrm{T}}_{L|ab} = -\eta\mathbf{\Sigma}_{L|ab}\eta, \quad \mathbf{\Sigma}^{\mathrm{T}}_{R|ab} = -\eta\mathbf{\Sigma}_{R|ab}\eta, \tag{9b}$$
$$\mathbf{\Sigma}^\dagger_{L|ab} = +\eta\mathbf{\Sigma}_{R|ab}\eta, \quad \mathbf{\Sigma}^\dagger_{R|ab} = +\eta\mathbf{\Sigma}_{L|ab}\eta. \tag{9c}$$

Also, they obey the Lie algebra

$$\mathrm{i}\left[\mathbf{\Sigma}_{X|ab}, \mathbf{\Sigma}_{X|cd}\right] = \eta_{ac}\mathbf{\Sigma}_{X|bd} - \eta_{ad}\mathbf{\Sigma}_{X|bc} - \eta_{bc}\mathbf{\Sigma}_{X|ad} + \eta_{bd}\mathbf{\Sigma}_{X|ac}, \tag{10}$$

so that $\mathbf{\Sigma}_{L|ab}$ and $\mathbf{\Sigma}_{R|ab}$ constitute generators of two different, but closely related (by complex conjugation), four-dimensional spin $\frac{1}{2}$ representations of $\mathrm{Spin}(3,1)$, because $\frac{1}{2}\mathbf{\Sigma}^{L|ij}\mathbf{\Sigma}_{L|ij} = \frac{1}{2}\mathbf{\Sigma}^{R|ij}\mathbf{\Sigma}_{R|ij} = \frac{3}{4}\mathbf{1}_4$.

C. Generators of $\mathrm{Spin}\,(3,1) \times \mathrm{SU}\,(2) \times \mathrm{U}\,(1)$

Eq. (5) implies that

$$\mathbf{0}_4 = \left[\mathbf{\Gamma}_{L|a}, \mathbf{\Sigma}_{R|cd}\right], \tag{11a}$$
$$\mathbf{0}_4 = \left[\mathbf{\Gamma}_{R|a}, \mathbf{\Sigma}_{L|cd}\right]. \tag{11b}$$

In conjunction with Eq. (5), these relations imply that (note the crossing of the L and R sectors) $\mathbf{\Sigma}_{L|cd}$ and $\mathbf{\Gamma}_{R|a}$ together, and $\mathbf{\Sigma}_{R|cd}$ and $\mathbf{\Gamma}_{L|a}$ together constitute two different, but closely related (by complex conjugation), four-dimensional representations of $\mathrm{Spin}\,(3,1) \times \mathrm{SU}\,(2) \times \mathrm{U}\,(1)$.

D. Generators of $\mathrm{SO}\,(3,1)$

Define the matrices $\mathbf{\Sigma}_{V|ab} \in \mathrm{M}\,(4,\mathbb{R})$ by

$$\mathrm{i}\left(\mathbf{\Sigma}_{V|ab}\right)_{cd} = \eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc}. \tag{12}$$

They obey the exact same Lie algebra as do $\mathbf{\Sigma}_{L|ab}$ and $\mathbf{\Sigma}_{R|ab}$, i.e., the Lie algebra given by Eq. (10) with $\mathbf{\Sigma}_{X|ab}$ replaced by $\mathbf{\Sigma}_{V|ab}$, so they constitute generators of the vector representation of $\mathrm{SO}\,(3,1)$, because $\frac{1}{2}\left(\mathbf{\Sigma}^{V|ij}\mathbf{\Sigma}_{V|ij}\right)_{kl} = 2\left(\mathbf{1}_3\right)_{kl}$.

The matrices $\mathbf{\Gamma}_{X|a}$ and $\mathbf{\Sigma}_{X|ab}$ are related to $\mathbf{\Sigma}_{V|ab}$ by the relations

$$-\left(\mathbf{\Sigma}_{V|ab}\right)^{c}{}_{d}\mathbf{\Gamma}^{X|d} = \mathbf{\Sigma}^{\dagger}_{X|ab}\mathbf{\Gamma}^{X|c} - \mathbf{\Gamma}^{X|c}\mathbf{\Sigma}_{X|ab}, \tag{13a}$$
$$+\left(\mathbf{\Sigma}_{V|ab}\right)^{d}{}_{c}\mathbf{\Gamma}_{X|d} = \mathbf{\Sigma}^{\dagger}_{X|ab}\mathbf{\Gamma}_{X|c} - \mathbf{\Gamma}_{X|c}\mathbf{\Sigma}_{X|ab}. \tag{13b}$$

E. Transformations and invariants

Define the matrices $\mathbf{\Lambda}_L, \mathbf{\Lambda}_R \in \mathrm{M}\,(4,\mathbb{C})$ and $\mathbf{\Lambda}_V \in \mathrm{M}\,(4,\mathbb{R})$ by

$$\mathbf{\Lambda}_L = \exp\left(-\frac{\mathrm{i}}{2}\theta^{ab}\mathbf{\Sigma}_{L|ab}\right),$$
$$\mathbf{\Lambda}_R = \exp\left(-\frac{\mathrm{i}}{2}\theta^{ab}\mathbf{\Sigma}_{R|ab}\right),$$
$$\mathbf{\Lambda}_V = \exp\left(-\frac{\mathrm{i}}{2}\theta^{ab}\mathbf{\Sigma}_{V|ab}\right),$$

where $\theta_{ab} = -\theta_{ba} \in \mathbb{R}$. From Eq. (9) it straightforwardly follows that

$$\mathbf{\Lambda}^{*}_L = \mathbf{\Lambda}_R, \quad \mathbf{\Lambda}^{*}_R = \mathbf{\Lambda}_L,$$
$$\mathbf{\Lambda}^{\mathrm{T}}_L = \eta\mathbf{\Lambda}^{-1}_L\eta, \quad \mathbf{\Lambda}^{\mathrm{T}}_R = \eta\mathbf{\Lambda}^{-1}_R\eta,$$
$$\mathbf{\Lambda}^{\dagger}_L = \eta\mathbf{\Lambda}^{-1}_R\eta, \quad \mathbf{\Lambda}^{\dagger}_R = \eta\mathbf{\Lambda}^{-1}_L\eta.$$

Remark Note that under transposition, $\mathbf{\Lambda}_L$ and $\mathbf{\Lambda}_R$ surprisingly behave exactly as does $\mathbf{\Lambda}_V$, which obeys $\mathbf{\Lambda}^{\mathrm{T}}_L = \eta\mathbf{\Lambda}^{-1}_V\eta$, the relation responsible for the invariance of the line element in the special theory of relativity.

From Eqs. (13) it follows that

$$\left(\mathbf{\Lambda}_V\right)^{a}{}_{b}\mathbf{\Gamma}^{X|b} = \mathbf{\Lambda}^{\dagger}_X\mathbf{\Gamma}^{X|a}\mathbf{\Lambda}_X,$$
$$\left(\mathbf{\Lambda}^{-1}_V\right)^{b}{}_{a}\mathbf{\Gamma}_{X|b} = \mathbf{\Lambda}^{\dagger}_X\mathbf{\Gamma}_{X|a}\mathbf{\Lambda}_X.$$

These relations imply, that if ψ_X are two four-spinors transforming as $\psi'_X = \mathbf{\Lambda}_X\psi_X$ under a Lorentz transformation, then $\psi^{\dagger}_X\mathbf{\Gamma}^{X|a}\partial_a\psi_X$ are invariants, because ∂_a transforms as $\partial'_a = \left(\mathbf{\Lambda}^{-1}_V\right)^{b}{}_{a}\partial_b$, and $\psi^{\dagger}_L\eta\psi_R$ and $\psi^{\dagger}_R\eta\psi_L$ (note the surprising appearance of η) are invariants.

IV. LAGRANGIAN

Consider the Lagrangian (note the explicit appearance of η in the mass terms)

$$\mathcal{L} = \mathbf{\Psi}^{\dagger}\begin{pmatrix} e^{\mu}{}_{a}\mathbf{\Gamma}^{L|a}\mathbf{D}_{L|\mu} & m^{*}\eta \\ m\eta & e^{\mu}{}_{a}\mathbf{\Gamma}^{R|a}\mathbf{D}_{R|\mu} \end{pmatrix}\mathbf{\Psi} + \mathrm{h.c.}, \tag{14}$$

where (note for the inner interactions $\mathbf{G}^{\mathrm{inner}}_{X|\mu}$ the crossing of the L and R sectors)

$$\mathbf{D}_{X|\mu} = \mathbf{1}_4\partial_\mu + \mathbf{G}^{\mathrm{outer}}_{X|\mu} + \mathbf{G}^{\mathrm{inner}}_{X|\mu},$$
$$\mathbf{G}^{\mathrm{outer}}_{X|\mu} = \frac{1}{2}\omega_{\mu}{}^{ab}\mathbf{\Sigma}_{X|ab},$$
$$\mathbf{G}^{\mathrm{inner}}_{L|\mu} = \frac{\mathrm{i}}{\hbar}gt_L W^{i}{}_{\mu}\left(\mathrm{i}\mathbf{\Gamma}_{R|i}\right) + \frac{\mathrm{i}}{\hbar}\frac{g'}{2}y_L B_\mu\left(\mathrm{i}\mathbf{\Gamma}_{R|0}\right),$$
$$\mathbf{G}^{\mathrm{inner}}_{R|\mu} = \frac{\mathrm{i}}{\hbar}gt_R W^{i}{}_{\mu}\left(\mathrm{i}\mathbf{\Gamma}_{L|i}\right) + \frac{\mathrm{i}}{\hbar}\frac{g'}{2}y_R B_\mu\left(\mathrm{i}\mathbf{\Gamma}_{L|0}\right).$$

The fields are: An eight-spinor field $\mathbf{\Psi}^{\mathrm{T}} \equiv \left(\psi^{\mathrm{T}}_L, \psi^{\mathrm{T}}_R\right)$, a vierbein field $e^{a}{}_{\mu}$ and its associated minimal spin connection $\omega_{\mu}{}^{ab} = g^{\rho\sigma}e^{a}{}_{\rho}\nabla_\mu e^{b}{}_{\sigma}$, see Ref. [2, Sec. 31.A], and $\mathrm{SU}\,(2)$ and $\mathrm{U}\,(1)$ gauge fields $W^{i}{}_{\mu}$ and B_μ, respectively. The constants are: Two real masses $m_1, m_2 \in \mathbb{R}$ combined into a complex mass $m \equiv m_1 + \mathrm{i}m_2$, coupling constants g and g' for $\mathrm{SU}\,(2)$ and $\mathrm{U}\,(1)$, respectively, and charges t_X and y_L for $\mathrm{SU}\,(2)$ and $\mathrm{U}\,(1)$, respectively, where $t_X = 0$ corresponds to an $\mathrm{SU}\,(2)$ singlet, and $t_X = \frac{1}{2}$ corresponds to an $\mathrm{SU}\,(2)$ doublet. The factor $\frac{1}{2}$ in connection with g' is present to be consistent with conventions [15, p. 428].

Due to the results of Sec. III, the Lagrangian is $\mathrm{Spin}\,(3,1) \times \mathrm{SU}\,(2) \times \mathrm{U}\,(1)$ gauge invariant, and it describes an eight-spinor field $\mathbf{\Psi}$ coupled to the external fields $\omega_{\mu}{}^{ab}$, and $W^{i}{}_{\mu}$ and B_μ.

Remark In Eq. (14), hermitian conjugation h.c. effectively applies to only the terms of the Lagrangian arising from $\mathbf{1}_4\partial_\mu$ and $\mathbf{G}^{\mathrm{outer}}_{X|\mu}$, because the terms arising from $\mathbf{G}^{\mathrm{inner}}_{X|\mu}$ are hermitian due to Eqs. (3c) and (5), and the mass terms that couple ψ_L and ψ_R are each others hermitian conjugate.

V. DISCUSSION

A notable feature of the Lagrangian, Eq. (14), is that the $\mathbf{\Gamma}_{X|a}$'s appearing in front of the (covariant) derivatives, as do the usual Dirac gamma matrices, also appear, although crossed in the L and R sense, in $\mathbf{G}^{\text{inner}}_{X|\mu}$ as $\mathrm{SU}(2)\times\mathrm{U}(1)$ generators. Is that profound?

Define the matrices $\mathbf{P}_{X|e}, \mathbf{P}_{X|\nu} \in \mathrm{M}(4,\mathbb{C})$ by (note the sign difference between the L and R sectors)

$$\begin{aligned}
\mathbf{P}_{L|e} &= -\frac{\mathrm{i}}{2}\left(\mathbf{\Gamma}_{R|0} - \mathbf{\Gamma}_{R|3}\right),\\
\mathbf{P}_{L|\nu} &= -\frac{\mathrm{i}}{2}\left(\mathbf{\Gamma}_{R|0} + \mathbf{\Gamma}_{R|3}\right),\\
\mathbf{P}_{R|e} &= -\frac{\mathrm{i}}{2}\left(\mathbf{\Gamma}_{L|0} + \mathbf{\Gamma}_{L|3}\right),\\
\mathbf{P}_{R|\nu} &= -\frac{\mathrm{i}}{2}\left(\mathbf{\Gamma}_{L|0} - \mathbf{\Gamma}_{L|3}\right).
\end{aligned}$$

They obey $\mathbf{P}^2_{X|e} = \mathbf{P}_{X|e}$ and $\mathbf{P}^2_{X|\nu} = \mathbf{P}_{X|\nu}$, and $\mathbf{1}_4 = \mathbf{P}_{X|e} + \mathbf{P}_{X|\nu}$ and $\mathbf{0}_4 = \mathbf{P}_{X|e}\mathbf{P}_{X|\nu} = \mathbf{P}_{X|\nu}\mathbf{P}_{X|e}$, so they are projection operators in the L and R sector, respectively. Because of Eqs. (4c) and (5), $\mathbf{P}_{L|e}$ and $\mathbf{P}_{L|\nu}$ commute with all terms in $\mathbf{D}_{L|\mu}$ except the terms arising from $\mathbf{\Sigma}_{L|ab}$, and $\mathbf{\Gamma}_{R|1}$ and $\mathbf{\Gamma}_{R|2}$. Analogously for $\mathbf{P}_{R|e}$ and $\mathbf{P}_{R|\nu}$. So, in the light of Eqs. (4a)-(4b) the matrices $\mathbf{P}_{X|e}$ and $\mathbf{P}_{X|\nu}$ may be considered weak isospin projection operators, a fact from which their subscripts e and ν, referring to the electron and neutrino, respectively, are derived from. Furthermore, because of the sign difference between the L and R sectors, most importantly (otherwise the mass terms would couple the different isospin components) they obey

$$\begin{aligned}
\mathbf{0}_4 &= \mathbf{P}_{L|e}\eta\mathbf{P}_{R|\nu} = \mathbf{P}_{L|\nu}\eta\mathbf{P}_{R|e},\\
\mathbf{0}_4 &= \mathbf{P}_{R|e}\eta\mathbf{P}_{L|\nu} = \mathbf{P}_{R|\nu}\eta\mathbf{P}_{L|e}.
\end{aligned}$$

Therefore, defining the four four-spinors $\psi_{X|e} = \mathbf{P}_{X|e}\psi_X$ and $\psi_{X|\nu} = \mathbf{P}_{X|\nu}\psi_X$, the mass terms of the Lagrangian may be written as

$$\mathrm{Re}\left(m^*\psi^\dagger_{L|e}\eta\psi_{R|e}\right) + \mathrm{Re}\left(m^*\psi^\dagger_{L|\nu}\eta\psi_{R|\nu}\right).$$

What significance, if any, is there to the explicit appearance of η, comparing it with the usual $2D$-block diagonal γ^0 in the mass term of the Dirac Lagrangian? Could the non-$2D$-block diagonal form of η, singling out one of four components, be connected with the missing component of the neutrino? And generally, is it any improvement that the usual Dirac projection operators $\frac{1}{2}\left(\mathbf{1}_4 \pm \gamma_5\right)$ are not present?

On a more speculative note, what happens when the complexified quaternions, here considered, is (almost irresistibly) generalized to the complexified octonions? Mathematically, there are some very compelling reasons for such a generalization:

1. The set of complexified quaternions is a natural subset of the set of complexified octonions, as the former can be embedded into the latter in numerous ways.

2. The proofs of Appendix B, with the sole exception being the proof of Eq. (5), which relies on associativity, a property the octonions does not have, carry over without any change for matrices $\mathbf{\Gamma}_{X|A}$ and $\mathbf{\Sigma}_{X|AB}$ (replacing $\mathbf{\Gamma}_{X|a}$ and $\mathbf{\Sigma}_{X|ab}$ considered in this article) defined by

$$\begin{aligned}
\left(\mathbf{\Gamma}_{L|A}\right)_{CD} &= \langle \mathrm{e}_C, \mathrm{e}_A\mathrm{e}_D\rangle,\\
\left(\mathbf{\Gamma}_{R|A}\right)_{CD} &= \langle \mathrm{e}_C, \mathrm{e}_D\mathrm{e}_A\rangle,
\end{aligned}$$

and [generators of the spinor representations of $\mathrm{Spin}(7,1)$]

$$\begin{aligned}
4\mathrm{i}\left(\mathbf{\Sigma}_{L|AB}\right)_{CD} &= \langle \mathrm{e}_A\mathrm{e}_C, \mathrm{e}_B\mathrm{e}_D\rangle - \langle \mathrm{e}_A\mathrm{e}_D, \mathrm{e}_B\mathrm{e}_C\rangle,\\
4\mathrm{i}\left(\mathbf{\Sigma}_{R|AB}\right)_{CD} &= \langle \mathrm{e}_C\mathrm{e}_A, \mathrm{e}_D\mathrm{e}_B\rangle - \langle \mathrm{e}_D\mathrm{e}_A, \mathrm{e}_C\mathrm{e}_B\rangle,
\end{aligned}$$

where $\mathrm{e}_A = (\mathrm{i}, \mathrm{e}_I) \in \mathbb{C}\otimes\mathbb{O}$ is a basis for the complexified octonions: e_I are the seven imaginary units of $\mathbb{O}$, obeying $\mathrm{e}_I\mathrm{e}_J = -\delta_{IJ} + {\psi_{IJ}}^K\mathrm{e}_K$, where ψ_{IJK} are the octonionic structure constants, see for instance Refs. [8, 9, 10, 11]. Of course, various other replacements must be made, for instance replacing $\eta \in \mathrm{M}(4,\mathbb{R})$ by the eight-dimensional Minkowski metric $\eta_8 \in \mathrm{M}(8,\mathbb{R})$. Might the requirement of associativity in the proof of Eq. (5), which holds for the complexified quaternions, but not for the complexified octonions, be the explanation for the four-dimensionality of spacetime, somehow forcing a $\mathbb{C}\otimes\mathbb{H}$-fibration of $\mathbb{C}\otimes\mathbb{O}$?

3. The quaternions and octonions share a *unique* property, although not utilized in this article: They allow the definition of triple cross products $X_L, X_R : (\mathbb{C}\otimes\mathbb{D})^3 \to \mathbb{C}\otimes\mathbb{D}$ (where $\mathbb{D}$ denotes either $\mathbb{H}$ or $\mathbb{O}$) by

$$\begin{aligned}
3!X_L(x,y,z) &= x\left(\overline{y}z - \overline{z}y\right) + \text{cyclic perm},\\
3!X_R(x,y,z) &= \left(x\overline{y} - y\overline{x}\right)z + \text{cyclic perm}.
\end{aligned}$$

The cross products X_L and X_R possess both the orthogonality property and the (generalized) Pythagorean property [6],

$$\begin{aligned}
0 &= \langle X(x_1,x_2,x_3), x_i\rangle,\\
\det\left(\langle x_i, x_j\rangle\right) &= \langle X(x_1,x_2,x_3), X(x_1,x_2,x_3)\rangle,
\end{aligned}$$

where the suppressed subscript means that the relations apply to both L and R. Trilinear cross products possessing both these properties exist *only* over algebras of real (or complex) dimension 4 or 8, see Refs. [6, 16], the underlying reason being the existence of precisely the division algebras $\mathbb{H}$ and $\mathbb{O}$.

4. The seemingly insignificant relation

$$\varepsilon_{abcd} = \mathrm{i}\langle X(\mathrm{e}_a, \mathrm{e}_b, \mathrm{e}_c), \mathrm{e}_d\rangle,$$

links duality in four-dimensional spacetime, as controlled by ε_{abcd}, with two natural structures of the (complex) quaternions, the inner product and the cross product, as defined above. This relation may be straightforwardly generalized to

$$\chi_{L|ABCD} = \mathrm{i}\left\langle X_L\left(\mathrm{e}_A, \mathrm{e}_B, \mathrm{e}_C\right), \mathrm{e}_D\right\rangle,$$
$$\chi_{R|ABCD} = \mathrm{i}\left\langle X_R\left(\mathrm{e}_A, \mathrm{e}_B, \mathrm{e}_C\right), \mathrm{e}_D\right\rangle,$$

where $\chi_{L|ABCD}$ and $\chi_{R|ABCD}$ are nonequal because of the nonassociativity of the (complexified) octonions. These structure constants $\chi_{L|ABCD}$ and $\chi_{R|ABCD}$ allow for the definition of self-duality in eight-dimensional spacetime of rank *two* tensors:

$$T_{AB} = \frac{\mathrm{i}}{2}\lambda_X \chi_{X|ABCD} T^{CD}.$$

It can be shown that the eigenvalues are $\lambda_L \in \{+1, -1/3\}$ and $\lambda_R \in \{-1, +1/3\}$. In the quaternionic case the eigenvalues are ± 1, as is well-known. Is the appearance of $\pm 1/3$ in the octonionic case somehow related to fractional (hyper)charges of the quarks?

It is the hope that some or all of these issues will be resolved in the near future.

APPENDIX A: IDENTITIES

The following Lemma lists some useful identities for composition algebras, a class to which the complexified quaternions belong, see for instance [13] or [14]. Note, though, that the normalization of the inner product in [13] and [14] differ by a factor of 2. The normalization used in Eq. (1) is the normalization used in [13]. However, [14] is mentioned because its overall presentation is clearer than that of [13], and as such may be valuable to the reader.

Lemma (See [13] or [14]) *The following identities hold for any composition algebra:*

$$\langle x, y\rangle \equiv \langle y, x\rangle, \tag{A1}$$
$$\langle x, y\rangle \equiv \langle \overline{x}, \overline{y}\rangle, \tag{A2}$$

and

$$\langle x, yz\rangle \equiv \langle \overline{y}x, z\rangle, \tag{A3}$$
$$\langle xy, z\rangle \equiv \langle x, z\overline{y}\rangle, \tag{A4}$$

and

$$x\left(\overline{y}z\right) + \left(y\overline{x}\right)z \equiv 2\langle x, y\rangle z, \tag{A5}$$
$$\left(x\overline{y}\right)z + \left(x\overline{z}\right)y \equiv 2\langle y, z\rangle x. \tag{A6}$$

APPENDIX B: PROOFS

Throughout this section the identities of the Lemma of Appendix A will be used without being explicitly referred to. Although the equations being proved below could reasonably simply be checked by explicit numerical calculation, by first calculating explicitly the four-dimensional matrices $\mathbf{\Gamma}_{L|a}$ and $\mathbf{\Gamma}_{R|a}$, and $\mathbf{\Sigma}_{L|ab}$ and $\mathbf{\Sigma}_{R|ab}$, using Eqs. (2) and (7), the main purpose of presenting analytical proofs is that the majority of these as stated, the sole exception being the proof of Eq. (5), apply to any composition algebra, and therefore in particular to both the complexified quaternions and the complexified octonions.

1. Proof of Eq. (3)

By direct calculation, using $\mathrm{e}_a^* = -\overline{\mathrm{e}}_a$ and $\overline{\mathrm{e}}_a = -\delta_{ab}\mathrm{e}^b$, respectively:

$$\begin{aligned}\left(\mathbf{\Sigma}_{L|a}^*\right)_{cd} &= \left[\left(\mathbf{\Sigma}_{L|a}\right)_{cd}\right]^* = \langle \mathrm{e}_c, \mathrm{e}_a\mathrm{e}_d\rangle^* = \langle \mathrm{e}_c^*, \mathrm{e}_a^*\mathrm{e}_d^*\rangle \\ &= -\langle \overline{\mathrm{e}}_c, \overline{\mathrm{e}}_a\overline{\mathrm{e}}_d\rangle = -\langle \mathrm{e}_c, \mathrm{e}_d\mathrm{e}_a\rangle = -\left(\mathbf{\Sigma}_{R|a}\right)_{cd},\end{aligned}$$

and

$$\begin{aligned}\left(\mathbf{\Sigma}_{L|a}^{\mathrm{T}}\right)^c{}_d &= \delta^{ce}\left(\mathbf{\Sigma}_{L|a}\right)^f{}_e\delta_{fd} = \delta^{ce}\left\langle \mathrm{e}^f, \mathrm{e}_a\mathrm{e}_e\right\rangle\delta_{fd} \\ &= \left\langle\left(-\delta_{df}\mathrm{e}^f\right), \mathrm{e}_a\left(-\delta^{ce}\mathrm{e}_e\right)\right\rangle = \langle \overline{\mathrm{e}}_d, \mathrm{e}_a\overline{\mathrm{e}}^c\rangle \\ &= \langle \overline{\mathrm{e}}_d\mathrm{e}^c, \mathrm{e}_a\rangle = \langle \mathrm{e}^c, \mathrm{e}_d\mathrm{e}_a\rangle = \left(\mathbf{\Sigma}_{R|a}\right)^c{}_d.\end{aligned}$$

The remaining assertion, Eq. (3c), readily follows from the matrix identity $\mathbf{M}^\dagger \equiv \left(\mathbf{M}^*\right)^{\mathrm{T}} \equiv \left(\mathbf{M}^{\mathrm{T}}\right)^*$.

2. Proof of Eq. (5)

Using the completeness relation $\langle x, \mathrm{e}_a\rangle\langle \mathrm{e}^a, y\rangle \equiv \langle x, y\rangle$:

$$\left[\mathbf{\Sigma}_{L|a}\right]^c{}_e\left[\mathbf{\Sigma}_{R|b}\right]^e{}_d = \langle \mathrm{e}^c, \mathrm{e}_a\mathrm{e}_e\rangle\langle \mathrm{e}^e, \mathrm{e}_d\mathrm{e}_b\rangle = \langle \overline{\mathrm{e}}_a\mathrm{e}^c, \mathrm{e}_d\mathrm{e}_b\rangle,$$
$$\left[\mathbf{\Sigma}_{R|b}\right]^c{}_e\left[\mathbf{\Sigma}_{L|a}\right]^e{}_d = \langle \mathrm{e}^c, \mathrm{e}_e\mathrm{e}_b\rangle\langle \mathrm{e}^e, \mathrm{e}_a\mathrm{e}_d\rangle = \langle \mathrm{e}^c\overline{\mathrm{e}}_b, \mathrm{e}_a\mathrm{e}_d\rangle.$$

These two expressions are equal because the (complexified) quaternions are *associative* (a property which breaks down when generalizing to complexified octonions) so that

$$\begin{aligned}\langle \overline{\mathrm{e}}_a\mathrm{e}^c, \mathrm{e}_d\mathrm{e}_b\rangle &= \langle \mathrm{e}^c, \mathrm{e}_a\left(\mathrm{e}_d\mathrm{e}_b\right)\rangle \\ &= \langle \mathrm{e}^c, \left(\mathrm{e}_a\mathrm{e}_d\right)\mathrm{e}_b\rangle \\ &= \langle \mathrm{e}^c\overline{\mathrm{e}}_b, \mathrm{e}_a\mathrm{e}_d\rangle.\end{aligned}$$

3. Proof of Eqs. (6) and (8)

Only the proof for L will be given, as the proof for R is completely analogous. Eq. (6) is proved as follows:

$$\begin{aligned}\left(\mathbf{\Sigma}^{\eta}_{L|a}\right)^{c}{}_{d} &= \left(\eta\mathbf{\Sigma}^{\mathrm{T}}_{L|a}\eta\right)^{c}{}_{d} = (\eta)^{c}{}_{e}\left(\mathbf{\Sigma}^{\mathrm{T}}_{L|a}\right)^{e}{}_{f}(\eta)^{f}{}_{d}\\ &= \eta^{ce}\left(\mathbf{\Sigma}_{L|a}\right)^{f}{}_{e}\eta_{fd} = \eta^{ce}\left\langle \mathrm{e}^{f}, \mathrm{e}_a\mathrm{e}_e\right\rangle\eta_{fd}\\ &= \left\langle \mathrm{e}_d, \mathrm{e}_a\mathrm{e}^{c}\right\rangle,\end{aligned}$$

which, using the completeness relation $\langle x, \mathrm{e}_a\rangle\langle \mathrm{e}^a, y\rangle \equiv \langle x, y\rangle$, implies that

$$\begin{aligned}\left(\mathbf{\Sigma}^{\eta}_{L|a}\mathbf{\Sigma}_{L|b}\right)^{c}{}_{d} &\equiv \left(\mathbf{\Sigma}^{\eta}_{L|a}\right)^{c}{}_{e}\left(\mathbf{\Sigma}_{L|b}\right)^{e}{}_{d}\\ &= \left\langle \mathrm{e}_e, \mathrm{e}_a\mathrm{e}^{c}\right\rangle\left\langle \mathrm{e}^{e}, \mathrm{e}_b\mathrm{e}_d\right\rangle\\ &= \left\langle \mathrm{e}_a\mathrm{e}^{c}, \mathrm{e}_b\mathrm{e}_d\right\rangle,\end{aligned}$$

which implies that

$$\begin{aligned}\left(\left[\mathbf{\Gamma}_{L|a}, \mathbf{\Gamma}_{L|b}\right]_{\eta+}\right)^{c}{}_{d} &= \left(\mathbf{\Sigma}^{\eta}_{L|a}\mathbf{\Sigma}_{L|b} + \mathbf{\Sigma}^{\eta}_{L|b}\mathbf{\Sigma}_{L|a}\right)^{c}{}_{d}\\ &= \left\langle \mathrm{e}_a\mathrm{e}^{c}, \mathrm{e}_b\mathrm{e}_d\right\rangle + \left\langle \mathrm{e}_b\mathrm{e}^{c}, \mathrm{e}_a\mathrm{e}_d\right\rangle\\ &= \left\langle \mathrm{e}_a, (\mathrm{e}_b\mathrm{e}_d)\,\overline{\mathrm{e}}^{c}\right\rangle + \left\langle (\mathrm{e}_b\mathrm{e}^{c})\,\overline{\mathrm{e}}_d, \mathrm{e}_a\right\rangle\\ &= \left\langle \mathrm{e}_a, (\mathrm{e}_b\mathrm{e}_d)\,\overline{\mathrm{e}}^{c} + (\mathrm{e}_b\mathrm{e}^{c})\,\overline{\mathrm{e}}_d\right\rangle\\ &= 2\left\langle \mathrm{e}_a, \mathrm{e}_b\right\rangle\left\langle \overline{\mathrm{e}}^{c}, \overline{\mathrm{e}}_d\right\rangle = 2\eta_{ab}\left(\mathbf{1}_4\right)^{c}{}_{d}.\end{aligned}$$

Eq. (8) follows directly from the second equation in the proof above for Eq. (6), and the defining equation of $\mathbf{\Sigma}_{L|ab}$ and $\mathbf{\Sigma}_{R|ab}$, Eq. (7).

4. Proof of Eq. (9)

Using Eqs. (3a) and (8):

$$\begin{aligned}-4\mathrm{i}\mathbf{\Sigma}^{*}_{L|ab} &= \left(\mathbf{\Gamma}^{\eta}_{L|a}\mathbf{\Gamma}_{L|b} - \mathbf{\Gamma}^{\eta}_{L|b}\mathbf{\Gamma}_{L|a}\right)^{*}\\ &= \eta\left(\mathbf{\Gamma}^{*}_{L|a}\right)^{\mathrm{T}}\eta\mathbf{\Gamma}^{*}_{L|b} - \eta\left(\mathbf{\Gamma}^{*}_{L|b}\right)^{\mathrm{T}}\eta\mathbf{\Gamma}^{*}_{L|a}\\ &= \eta\mathbf{\Gamma}^{\mathrm{T}}_{R|a}\eta\mathbf{\Gamma}_{R|b} - \eta\mathbf{\Gamma}^{\mathrm{T}}_{R|b}\eta\mathbf{\Gamma}_{R|a}\\ &= 4\mathrm{i}\mathbf{\Sigma}_{R|ab}.\end{aligned}$$

Using Eq. (8):

$$\begin{aligned}4\mathrm{i}\mathbf{\Sigma}^{\mathrm{T}}_{X|ab} &= \left(\mathbf{\Gamma}^{\eta}_{X|a}\mathbf{\Gamma}_{X|b} - \mathbf{\Gamma}^{\eta}_{X|b}\mathbf{\Gamma}_{X|a}\right)^{\mathrm{T}}\\ &= \mathbf{\Gamma}^{\mathrm{T}}_{X|b}\eta\mathbf{\Gamma}_{X|a}\eta - \mathbf{\Gamma}^{\mathrm{T}}_{X|a}\eta\mathbf{\Gamma}_{X|b}\eta\\ &= \eta\left(\mathbf{\Gamma}^{\eta}_{X|b}\mathbf{\Gamma}_{X|a} - \mathbf{\Gamma}^{\eta}_{X|a}\mathbf{\Gamma}_{X|b}\right)\eta\\ &= -4\mathrm{i}\eta\mathbf{\Sigma}_{X|ab}\eta.\end{aligned}$$

The remaining assertion, Eq. (9c), readily follows from the matrix identity $\mathbf{M}^{\dagger} \equiv \left(\mathbf{M}^{*}\right)^{\mathrm{T}} \equiv \left(\mathbf{M}^{\mathrm{T}}\right)^{*}$.

5. Proof of Eq. (10)

To compactify the calculations, the subscript $X|$ has been dropped throughout. Consider the expression $\mathbf{\Gamma}^{\eta}_a\mathbf{\Gamma}_b\mathbf{\Gamma}^{\eta}_c\mathbf{\Gamma}_d$. Using fourfoldly Eq. (6) to move $\mathbf{\Gamma}^{\eta}_a\mathbf{\Gamma}_b$ through $\mathbf{\Gamma}^{\eta}_c\mathbf{\Gamma}_d$, it follows that

$$\begin{aligned}\left[\mathbf{\Gamma}^{\eta}_a\mathbf{\Gamma}_b, \mathbf{\Gamma}^{\eta}_c\mathbf{\Gamma}_d\right] &= 2\eta_{ac}\mathbf{\Gamma}^{\eta}_d\mathbf{\Gamma}_b - 2\eta_{ad}\mathbf{\Gamma}^{\eta}_c\mathbf{\Gamma}_b\\ &\quad + 2\eta_{bc}\mathbf{\Gamma}^{\eta}_a\mathbf{\Gamma}_d - 2\eta_{bd}\mathbf{\Gamma}^{\eta}_a\mathbf{\Gamma}_c.\end{aligned}$$

Using fourfoldly this result in the expression

$$\begin{aligned}-16\left[\mathbf{\Sigma}_{ab}, \mathbf{\Sigma}_{cd}\right] &= \left[\left[\mathbf{\Gamma}_a, \mathbf{\Gamma}_b\right]_{\eta-}, \left[\mathbf{\Gamma}_c, \mathbf{\Gamma}_d\right]_{\eta-}\right]\\ &= \left[\mathbf{\Gamma}^{\eta}_a\mathbf{\Gamma}_b, \mathbf{\Gamma}^{\eta}_c\mathbf{\Gamma}_d\right] - \left[\mathbf{\Gamma}^{\eta}_a\mathbf{\Gamma}_b, \mathbf{\Gamma}^{\eta}_d\mathbf{\Gamma}_c\right]\\ &\quad - \left[\mathbf{\Gamma}^{\eta}_b\mathbf{\Gamma}_a, \mathbf{\Gamma}^{\eta}_c\mathbf{\Gamma}_d\right] + \left[\mathbf{\Gamma}^{\eta}_b\mathbf{\Gamma}_a, \mathbf{\Gamma}^{\eta}_d\mathbf{\Gamma}_c\right],\end{aligned}$$

collecting identical terms, and using again Eq. (6), yields

$$\begin{aligned}-16\left[\mathbf{\Sigma}_{ab}, \mathbf{\Sigma}_{cd}\right] &= -4\eta_{ac}\left[\mathbf{\Gamma}_b, \mathbf{\Gamma}_d\right]_{\eta-} + 4\eta_{ad}\left[\mathbf{\Gamma}_b, \mathbf{\Gamma}_c\right]_{\eta-}\\ &\quad + 4\eta_{bc}\left[\mathbf{\Gamma}_a, \mathbf{\Gamma}_d\right]_{\eta-} - 4\eta_{bd}\left[\mathbf{\Gamma}_a, \mathbf{\Gamma}_c\right]_{\eta-},\end{aligned}$$

from which the result follows.

Remark The proof is completely analogous to the proof of the assertion that $-\frac{\mathrm{i}}{4}\left[\gamma_a, \gamma_b\right]$, where γ_a are the Dirac matrices obeying $2\eta_{ab}\mathbf{1}_4 = \{\gamma_a, \gamma_b\}$, are generators of $\mathrm{Spin}\,(3,1)$. That is the main reason for introducing above the (anti)commutator-like brackets $[\cdot,\cdot]_{\eta\pm}$.

6. Proof of Eq. (11)

Only the proof of Eq. (11a) will be given, as the proof of Eq. (11b) is completely analogous. Using $\overline{\mathrm{e}}_a = -\delta_{ab}\mathrm{e}^{b}$:

$$\begin{aligned}\left(\mathbf{\Sigma}^{\eta}_{L|a}\right)^{c}{}_{d} &= \left(\eta\mathbf{\Sigma}^{\mathrm{T}}_{L|a}\eta\right)^{c}{}_{d} = (\eta)^{c}{}_{e}\left(\mathbf{\Sigma}^{\mathrm{T}}_{L|a}\right)^{e}{}_{f}(\eta)^{f}{}_{d}\\ &= \eta^{ce}\left(\mathbf{\Sigma}_{L|a}\right)^{f}{}_{e}\eta_{fd} = \eta^{ce}\left\langle \mathrm{e}^{f}, \mathrm{e}_a\mathrm{e}_e\right\rangle\eta_{fd}\\ &= \left\langle \mathrm{e}_d, \mathrm{e}_a\mathrm{e}^{c}\right\rangle = \left\langle \overline{\mathrm{e}}_d, \overline{\mathrm{e}}^{c}\overline{\mathrm{e}}_a\right\rangle = \left\langle \mathrm{e}^{c}, \overline{\mathrm{e}}_a\mathrm{e}_d\right\rangle\\ &= -\delta_{ab}\left\langle \mathrm{e}^{c}, \overline{\mathrm{e}}^{b}\mathrm{e}_d\right\rangle = -\delta_{ab}\left(\mathbf{\Sigma}^{L|b}\right)^{c}{}_{d},\end{aligned}$$

which, using Eq. (5), implies that

$$\begin{aligned}4\mathrm{i}\left[\mathbf{\Gamma}_{L|a}, \mathbf{\Sigma}_{R|cd}\right] &= \left[\mathbf{\Gamma}_{L|a}, \left[\mathbf{\Gamma}_{R|c}, \mathbf{\Gamma}_{R|d}\right]_{\eta-}\right]\\ &= \mathbf{\Gamma}_{L|a}\mathbf{\Gamma}^{\eta}_{R|c}\mathbf{\Gamma}_{R|d} - \mathbf{\Gamma}_{L|a}\mathbf{\Gamma}^{\eta}_{R|d}\mathbf{\Gamma}_{R|c}\\ &\quad - \mathbf{\Gamma}^{\eta}_{R|c}\mathbf{\Gamma}_{L|a}\mathbf{\Gamma}_{R|d} + \mathbf{\Gamma}^{\eta}_{R|d}\mathbf{\Gamma}_{L|a}\mathbf{\Gamma}_{R|c}\\ &= \left(\mathbf{\Gamma}_{L|a}\mathbf{\Gamma}^{\eta}_{R|c} - \mathbf{\Gamma}^{\eta}_{R|c}\mathbf{\Gamma}_{L|a}\right)\mathbf{\Gamma}_{R|d}\\ &\quad - \left(\mathbf{\Gamma}_{L|a}\mathbf{\Gamma}^{\eta}_{R|d} - \mathbf{\Gamma}^{\eta}_{R|d}\mathbf{\Gamma}_{L|a}\right)\mathbf{\Gamma}_{R|c}\\ &= -\delta_{ce}\left[\mathbf{\Gamma}_{L|a}, \mathbf{\Gamma}^{R|e}\right]\mathbf{\Gamma}_{R|d}\\ &\quad + \delta_{de}\left[\mathbf{\Gamma}_{L|a}, \mathbf{\Gamma}^{R|e}\right]\mathbf{\Gamma}_{R|c} = \mathbf{0}_4.\end{aligned}$$

7. Proof of Eq. (13)

Only the proof of Eq. (13a) will be given, as the proof of Eq. (13b) is analogous. Using Eq. (3c):

$$\left(\mathbf{\Gamma}^{\eta}_{X|a}\mathbf{\Gamma}_{X|b}\right)^{\dagger} = \mathbf{\Gamma}^{\dagger}_{X|b}\left(\mathbf{\Gamma}^{\dagger}_{X|a}\right)^{\eta} = \mathbf{\Gamma}_{X|b}\mathbf{\Gamma}^{\eta}_{X|a},$$

which, using Eqs. (6) and (8), implies that (where to compactify the calculations, the subscript $X|$ has been dropped throughout)

$$\begin{aligned}
4\mathrm{i}\left(\mathbf{\Sigma}^{\dagger}_{ab}\mathbf{\Gamma}^{c} - \mathbf{\Gamma}^{c}\mathbf{\Sigma}_{ab}\right) &= \mathbf{\Gamma}_{a}\left[2\delta^{c}_{b}\mathbf{1}_{4} - \left(\mathbf{\Gamma}^{c}\right)^{\eta}\mathbf{\Gamma}_{b}\right] - \mathbf{\Gamma}^{c}\mathbf{\Gamma}^{\eta}_{a}\mathbf{\Gamma}_{b} \\
&\quad - \mathbf{\Gamma}_{b}\left[2\delta^{c}_{a}\mathbf{1}_{4} - \left(\mathbf{\Gamma}^{c}\right)^{\eta}\mathbf{\Gamma}_{a}\right] + \mathbf{\Gamma}^{c}\mathbf{\Gamma}^{\eta}_{b}\mathbf{\Gamma}_{a} \\
&= 2\delta^{c}_{b}\mathbf{\Gamma}_{a} - \left[\mathbf{\Gamma}_{a}\left(\mathbf{\Gamma}^{c}\right)^{\eta} + \mathbf{\Gamma}^{c}\mathbf{\Gamma}^{\eta}_{a}\right]\mathbf{\Gamma}_{b} \\
&\quad - 2\delta^{c}_{a}\mathbf{\Gamma}_{b} + \left[\mathbf{\Gamma}_{b}\left(\mathbf{\Gamma}^{c}\right)^{\eta} + \mathbf{\Gamma}^{c}\mathbf{\Gamma}^{\eta}_{b}\right]\mathbf{\Gamma}_{a} \\
&= 2\delta^{c}_{b}\mathbf{\Gamma}_{a} - \left[\mathbf{\Gamma}^{c}, \mathbf{\Gamma}_{a}\right]^{\dagger}_{\eta+}\mathbf{\Gamma}_{b} \\
&\quad - 2\delta^{c}_{a}\mathbf{\Gamma}_{b} + \left[\mathbf{\Gamma}^{c}, \mathbf{\Gamma}_{b}\right]^{\dagger}_{\eta+}\mathbf{\Gamma}_{a} \\
&= 4\left(\delta^{c}_{b}\eta_{ad} - \delta^{c}_{a}\eta_{bd}\right)\mathbf{\Gamma}^{d} \\
&= -4\mathrm{i}\left(\mathbf{\Sigma}_{V|ab}\right)^{c}{}_{d}\mathbf{\Gamma}^{d}.
\end{aligned}$$

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